

A New Supersymmetric Extension of Conformal Mechanics.

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Abstract

In this paper we propose a new supersymmetric extension of conformal mechanics. The Grassmannian variables that we introduce are the basis of the forms and of the vector-fields built over the symplectic space of the original system. Our supersymmetric Hamiltonian itself turns out to have a clear geometrical meaning being the Lie-derivative of the Hamiltonian flow of conformal mechanics. Using superfields we derive a constraint which gives the exact solution of the supersymmetric system in a way analogous to the constraint in configuration space which solved the original non-supersymmetric model. Besides the supersymmetric extension of the original Hamiltonian, we also provide the extension of the other conformal generators present in the original system. These extensions have also a supersymmetric character being the square of some Grassmannian charge. We build the whole superalgebra of these charges and analyze their closure. The representation of the even part of this superalgebra on the odd part turns out to be integer and not spinorial in character.

1 Introduction

More than 20 years ago a conformally-invariant quantum mechanical model was proposed and studied in ref.[1]. Recently that model has attracted again some interest in connection with black-holes. It has been proven [2] in fact that the dynamics of a particle near the horizon of an extreme Reissner-Nordström black-hole is governed in its radial motion by the Lagrangian of ref. [1]. This can be considered a manifestation at the quantum mechanical level of the correspondence between Gravity on AdS and conformal field theory [3]. In this case it is the correspondence AdS_2/CFT_1 .

In 1983/1984 two groups [4] independently made a supersymmetric generalization of conformal mechanics. It has been proved recently [2] that the motion of a *super*-particle near an extreme Reissner-Nordström black-hole is governed by a “relativistic” generalization of the supersymmetric conformal mechanics proposed in ref. [4]. So also the supersymmetric version of conformal mechanics seems to hold some interest for black-hole physics. For these conformal models (both for the original [1] and for the supersymmetric one [4]) a nice geometrical approach was pioneered in ref. [5].

In this paper we will build a new supersymmetric extension of conformal mechanics. It is tailored on a path-integral approach to classical mechanics developed in ref. [6]. The Grassmannian variables which appear in this formalism are the basis of the forms [7] and vector fields which one can build over the symplectic space of the original conformal system. The weight in this path-integral provides what is known as the Lie-derivative [7] of the Hamiltonian flow. This Lie-derivative turns out to be supersymmetric [6].

The reader may ask which is the difference between our extension and the one of ref. [4] which was tailored on the supersymmetric quantum mechanics of Witten [8]. Basically the authors of ref. [4] took the original conformal Hamiltonian and added a Grassmannian part in order to make the whole Hamiltonian supersymmetric. Our procedure and extension is different and more geometrical as will be explained later on in the paper. One difference for example is that while the equations of motion for our bosonic variables are the same as those of the old conformal model [1], the analog equations of ref. [4] have an extra piece. Another difference is that, once we use the language of superfields, while we have to stick the superfield into the old conformal potential [1] the authors of ref. [4] have to stick it into a potential which is *derived* from the old conformal potential.

The paper is organized in the following manner: In section **2** we give a very brief outline of conformal mechanics [1] and of the supersymmetric extension present in the literature [4]; in section **3** we put forward our supersymmetric extension and explain its geometrical structure. In the same section we build a whole set of charges connected with our extension and study their algebra in details. In section **4** we show that,

differently from the superconformal algebra of [4] where the even part has a spinorial representation on the odd part, ours is a non-simple superalgebra whose even part has a reducible and integer representation on the odd part. Nevertheless we can call it a superconformal algebra because we have the charges which result from a combined supersymmetry and conformal transformation. We think that the usual idea [12] which says that a superconformal algebra has its even part represented spinorially on the odd one applies only to relativistic systems and not to ours. In section 5 we provide a superspace version for our model and for the whole set of charges. In doing that we find a new *universal* charge which is present always in our formalism [6] even for non-conformal model. This new charge acts on the variables in such a manner as to rescale the over-all Lagrangian. This charge is not present in supersymmetric quantum mechanics [4]. In section 6, using superfield variables, we provide an exact solution of our model by giving a constraint in superfield space analogous to the one found for configuration space in [1] and which provided the solution of the original conformal system. We confine some calculations to a couple of appendices.

In this paper we do not have applications of our supersymmetric extension of conformal mechanics to black-hole physics. We thought anyhow worth to write up these mathematical results because we feel that something deeply geometrical is behind the connection conformal mechanics/black-holes or in general the correspondence *AdS/CFT*. To grasp these geometrical issues it is better to use models where geometry is manifest and beautiful and we believe ours has these features. We leave to others the task of finding possible connection of our supersymmetric extension with black-holes.

2 Conformal Mechanics and its Supersymmetric Extension

In this section we will briefly review conformal mechanics [1] and its standard supersymmetric extension [4].

The Lagrangian proposed in [1] is

$$L = \frac{1}{2} \left[\dot{q}^2 - \frac{g}{q^2} \right]. \quad (1)$$

It is easy to prove that this Lagrangian is invariant under the following transformations:

$$t' = \frac{\alpha t + \beta}{\gamma t + \delta}; \quad (2)$$

$$q'(t') = \frac{q(t)}{(\gamma t + \delta)}; \quad (3)$$

$$\text{with} \quad \alpha\delta - \beta\gamma = 1; \quad (4)$$

which are nothing else than the conformal transformations in 0+1 dimensions. They are made of the combinations of the following three transformations:

$$t' = \alpha^2 t \quad \text{dilations}, \quad (5)$$

$$t' = t + \beta \quad \text{time-translations}, \quad (6)$$

$$t' = \frac{t}{\gamma t + 1} \quad \text{special-conformal transformations}. \quad (7)$$

The Noether charges associated [1] to these three symmetries are:

$$H = \frac{1}{2} \left(p^2 + \frac{g}{q^2} \right); \quad (8)$$

$$D = tH - \frac{1}{4}(qp + pq); \quad (9)$$

$$K = t^2 H - \frac{1}{2}t(qp + pq) + \frac{1}{2}q^2. \quad (10)$$

Using the quantum commutator $[q, p] = i$, the algebra of the three Noether charges above is:

$$[H, D] = iH; \quad (11)$$

$$[K, D] = -iK; \quad (12)$$

$$[H, K] = 2iD. \quad (13)$$

The fact that D and K do not commute with H does not mean that they are not conserved. In fact what is true is that, as the H, D, K above are explicitly dependent on t , we have:

$$\frac{\partial D}{\partial t} \neq 0; \quad \frac{\partial K}{\partial t} \neq 0; \quad (14)$$

$$\frac{dD}{dt} = \frac{dK}{dt} = 0. \quad (15)$$

As the H, D, K are conserved, their expressions at $t = 0$ which are¹:

$$H_0 = \frac{1}{2} \left[p^2 + \frac{g}{q^2} \right], \quad (16)$$

$$D_0 = -\frac{1}{4} [qp + pq], \quad (17)$$

$$K_0 = \frac{1}{2} q^2, \quad (18)$$

satisfy the same algebra as those at time t (see eqs. (11)-(13)). This algebra is $SO(2, 1)$ which is known [11] to be isomorphic to the conformal group in $0 + 1$ dimensions.

Let us now turn to the supersymmetric extension of this model proposed in ref. [4]. The Hamiltonian is:

$$H_{SUSY} = \frac{1}{2} \left(p^2 + \frac{g}{q^2} + \frac{\sqrt{g}}{q^2} [\psi^\dagger, \psi]_- \right) \quad (19)$$

where ψ, ψ^\dagger are Grassmannian variables whose anticommutator is $[\psi, \psi^\dagger]_+ = 1$. As one can notice, in H_{SUSY} there is a first bosonic piece which is the conformal Hamiltonian of eq.(1), plus a Grassmannian part. Note that the equations of motion for “ q ” have an extra piece with respect to the equations of motion of the old conformal mechanics [1].

To make contact with supersymmetric quantum mechanics [8] let us notice that H_{SUSY} can be written as:

$$H_{SUSY} = \frac{1}{2} [Q, Q^\dagger]_+ = \quad (20)$$

$$= \frac{1}{2} \left(p^2 + \left(\frac{dW}{dq} \right)^2 - [\psi^\dagger, \psi]_- \frac{d^2 W}{dq^2} \right) \quad (21)$$

where the supersymmetry charges are:

$$Q = \psi^\dagger \left(-ip + \frac{dW}{dq} \right), \quad (22)$$

$$Q^\dagger = \psi \left(ip + \frac{dW}{dq} \right), \quad (23)$$

¹The RHS of eqs.(16)-(18) is understood with p and q at $t = 0$ even if we do not put any subindex $(.)_0$ on them. Moreover H_0 and H have even the same functional form and so we will drop the subindex $(.)_0$ on H .

and W is the superpotential which, in this case of conformal mechanics, turns out to be:

$$W(q) = \sqrt{g} \log q. \quad (24)$$

It is interesting to see what we obtain when we combine a supersymmetric transformation with a conformal one generated by the (H, K, D) elements of the $SO(2, 1)$ algebra (11),(12),(13). We get what is called a *superconformal* transformation. In order to understand this better let us list the following eight operators:

TABLE 1

H	$= \frac{1}{2} \left[p^2 + \frac{g + 2\sqrt{g}B}{q^2} \right];$
D	$= -\frac{[q, p]_+}{4};$
K	$= \frac{q^2}{2};$
B	$= \frac{[\psi^\dagger, \psi]_-}{2};$
Q	$= \psi^\dagger \left[-ip + \frac{\sqrt{g}}{q} \right];$
Q^\dagger	$= \psi \left[ip + \frac{\sqrt{g}}{q} \right];$
S	$= \psi^\dagger q;$
S^\dagger	$= \psi q.$

The algebra of these operators is closed and given in the table below:

TABLE 2

$[H, D] = iH;$	$[K, D] = -iK;$	$[H, K] = 2iD;$
$[Q, H] = 0;$	$[Q^\dagger, H] = 0;$	$[Q, D] = \frac{i}{2}Q;$
$[Q^\dagger, K] = S^\dagger;$	$[Q, K] = -S;$	$[Q^\dagger, D] = \frac{i}{2}Q^\dagger;$
$[S, K] = 0;$	$[S^\dagger, K] = 0;$	$[S, D] = -\frac{i}{2}S;$
$[S^\dagger, D] = -\frac{i}{2}S^\dagger;$	$[S, H] = -Q;$	$[S^\dagger, H] = Q^\dagger;$
$[Q, Q^\dagger] = 2H;$	$[S, S^\dagger] = 2K;$	
$[B, S] = S;$	$[B, S^\dagger] = -S^\dagger;$	
$[Q, S^\dagger] = \sqrt{g} - B + 2iD;$	$[B, Q] = Q;$	$[B, Q^\dagger] = -Q^\dagger;$

all other commutators are zero or derivable from these by Hermitian conjugation. The square-brackets $[(\cdot), (\cdot)]$ in the algebra above are *graded*-commutators and from now on we shall not put on them the subindex $+$ or $-$ as we did before. They are commutators or anticommutators according to the Grassmannian nature of the operators entering the brackets.

As it is well known a *superconformal* transformation is a combination of a supersymmetry transformation and a conformal one. We see from the algebra above that the commutators of the supersymmetry generators (Q, Q^\dagger) with the three conformal generators (H, K, D) generate a new operator which is S . Including this new one we generate an algebra which is closed provided that we introduce the operator B of **TABLE 3**. This is the last operator we need.

3 A New Supersymmetric Extension of Conformal Mechanics

In this section we are going to present a new supersymmetric extension of conformal mechanics. This extension is tailored on a path-integral approach to classical mechanics (CM) developed in ref. [6]. The idea is to give a *path integral* for CM whose operatorial counterpart be the well-known *operatorial* version of CM as given by the *Liouville* operator [9]. We will be brief here because more details can be found in [6].

Let us start with a $2n$ -dimensional phase space \mathcal{M} whose coordinates are indicated as ϕ^a with $a = 1, \dots, 2n$, i.e.: $\phi^a = (q^1, \dots, q^n; p^1, \dots, p^n)$. Let us write the Hamiltonian of the system as $H(\phi)$ and the symplectic-matrix as ω^{ab} . The equations of motion are then:

$$\dot{\phi}^a = \omega^{ab} \frac{\partial H}{\partial \phi^b}. \quad (25)$$

We shall put forward, as path integral for CM, one that forces all paths in \mathcal{M} to sit on the classical ones. The *classical* analog of the quantum generating functional is:

$$Z_{CM}[J] = N \int \mathcal{D}\phi \, \tilde{\delta}[\phi(t) - \phi_{cl}(t)] \exp \left[\int J \phi \, dt \right] \quad (26)$$

where ϕ are the $\phi^a \in \mathcal{M}$, ϕ_{cl} are the solutions of eq.(25), J is an external current and $\tilde{\delta}[\cdot]$ is a functional Dirac delta which forces every path $\phi(t)$ to sit on a classical one $\phi_{cl}(t)$. Of course there are all possible initial conditions integrated over in eq.(26).

We should now check if the path integral of eq.(26) leads to the operatorial formulation [9] of CM. To do that let us first rewrite the functional Dirac delta in eq.(26) as:

$$\tilde{\delta}[\phi - \phi_{cl}] = \tilde{\delta}[\dot{\phi}^a - \omega^{ab}\partial_b H] \det[\delta_b^a \partial_t - \omega^{ac}\partial_c \partial_b H] \quad (27)$$

where we have used the functional analog of the relation $\delta[f(x)] = \frac{\delta[x-x_i]}{\left|\frac{\partial f}{\partial x}\right|_{x_i}}$. The determinant which appears in eq.(27) is always positive and so we can drop the modulus sign $|\cdot|$. The next step is to insert eq.(27) in eq.(26) and write the $\tilde{\delta}[\cdot]$ as a Fourier transform over some new variables λ_a , i.e.:

$$\tilde{\delta}\left[\dot{\phi}^a - \omega^{ab}\frac{\partial H}{\partial \phi^b}\right] = \int \mathcal{D}\lambda_a \exp\left\{i \int \lambda_a \left[\dot{\phi}^a - \omega^{ab}\frac{\partial H}{\partial \phi^b}\right] dt\right\}. \quad (28)$$

We then express the determinant $\det[\delta_b^a \partial_t - \omega^{ac}\partial_c \partial_b H]$ via Grassmannian variables \bar{c}_a, c^a as:

$$\det[\delta_b^a \partial_t - \omega^{ac}\partial_c \partial_b H] = \int \mathcal{D}c^a \mathcal{D}\bar{c}_a \exp\left\{-\int \bar{c}_a [\delta_b^a \partial_t - \omega^{ac}\partial_c \partial_b H] c^b dt\right\}. \quad (29)$$

Inserting the RHS of eqs. (28)(29) in eq.(27) and then in eq.(26) we get:

$$Z_{CM}[0] = \int \mathcal{D}\phi^a \mathcal{D}\lambda_a \mathcal{D}c^a \mathcal{D}\bar{c}_a \exp\left[i \int dt \tilde{\mathcal{L}}\right] \quad (30)$$

where $\tilde{\mathcal{L}}$ is:

$$\tilde{\mathcal{L}} = \lambda_a [\dot{\phi}^a - \omega^{ab}\partial_b H] + i\bar{c}_a [\delta_b^a \partial_t - \omega^{ac}\partial_c \partial_b H] c^b. \quad (31)$$

One can easily see that this Lagrangian gives the following equations of motion: for ϕ and c :

$$\dot{\phi}^a - \omega^{ab}\partial_b H = 0 \quad (32)$$

$$[\delta_b^a \partial_t - \omega^{ac}\partial_c \partial_b H] c^b = 0. \quad (33)$$

One notices immediately the following two things:

- 1) $\tilde{\mathcal{L}}$ leads to the same Hamiltonian equations for ϕ as H did;
- 2) c^b transforms under the Hamiltonian vector field $h \equiv \omega^{ab}\partial_b H \partial_a$ as a *form* $d\phi^b$ does.

From the above formalism, using some extended Poisson brackets (*EPB*) defined in the space $(\phi^a, c^a, \lambda_a, \bar{c}_a)$, one can get the equations of motion also via an Hamiltonian $\tilde{\mathcal{H}}$ given by

$$\tilde{\mathcal{H}} = \lambda_a \omega^{ab} \partial_b H + i\bar{c}_a \omega^{ac} (\partial_c \partial_b H) c^b. \quad (34)$$

The extended Poisson brackets mentioned above are:

$$\{\phi^a, \lambda_b\}_{EPB} = \delta_b^a; \quad \{\bar{c}_b, c^a\}_{EPB} = -i\delta_b^a. \quad (35)$$

The equations of motion are then given by $\frac{d}{dt}A = \{A, \widetilde{\mathcal{H}}\}_{EPB}$ where A is one of the variables $(\phi^a, c^a, \lambda_a, \bar{c}_a)$. All the other EPB are zero; in particular $\{\phi^a, \phi^b\}_{EPB} = 0$. This indicates that the EPB are not the standard Poisson brackets on \mathcal{M} which would give $\{\phi^a, \phi^b\}_{PB} = \omega^{ab}$.

Being eq.(30) a path integral one could also introduce the concept of *commutator* as Feynman did in the quantum case. If we define the graded commutator of two functions $O_1(t)$ and $O_2(t)$ as the expectation value $\langle \dots \rangle$ under our path integral of some time-splitting combinations of the functions themselves as:

$$\langle [O_1(t), O_2(t)] \rangle \equiv \lim_{\epsilon \rightarrow 0} \langle O_1(t + \epsilon) O_2(t) \pm O_2(t + \epsilon) O_1(t) \rangle, \quad (36)$$

then we get from eq.(30) that the only commutators different from zero among the basic variables are:

$$\langle [\phi^a, \lambda_b] \rangle = i\delta_b^a; \quad \langle [\bar{c}_b, c^a] \rangle = \delta_b^a. \quad (37)$$

We notice immediately two things:

- A)** there is an isomorphism between the extended Poisson structure (35) and the graded commutator structure (37): $\{.,.\}_{EPB} \longrightarrow -i[.,.];$
- B)** via the commutator structure (37) one can “realize” λ_a and \bar{c}_a as:

$$\lambda_a = -i\frac{\partial}{\partial\phi^a}; \quad \bar{c}_a = \frac{\partial}{\partial c^a}. \quad (38)$$

It is now easy to check that, using eq.(38), what we got as weight in eq.(30) corresponds to the operatorial version of CM. In fact take for the moment only the bosonic part of $\widetilde{\mathcal{H}}$ in eq.(34): $\widetilde{\mathcal{H}}_{bos} = \lambda_a \omega^{ab} \partial_b H$; this one, via eq.(38), goes into the operator $\widetilde{\widetilde{\mathcal{H}}}_{bos} \equiv -i\omega^{ab} \partial_b H \partial_a$ which is nothing else than the Liouville operator of CM. So we got what we expected. If we had added the Grassmannian part to $\widetilde{\mathcal{H}}_{bos}$ and inserted the operatorial representation (38) of \bar{c} , we would have got an operator which makes the evolution of functions depending not only on ϕ but also on c like $F(\phi, c) = F_{a_1 \dots a_p} c^{a_1} \dots c^{a_p}$. Remembering that the c^a transform as $d\phi^a$ (see point **(1)** below eq.(33)), we can say that the function $F(\phi, c)$ can be put in correspondence with p -forms:

$$F = F_{a_1 \dots a_p} c^{a_1} \dots c^{a_p} \longrightarrow F_{a_1 \dots a_p} d\phi^{a_1} \wedge \dots \wedge d\phi^{a_p}. \quad (39)$$

So our $\widetilde{\mathcal{H}}$ makes the evolution of forms that means it corresponds to the object known in literature [7] as the Lie-Derivative of the Hamiltonian flow. Note that we are not talking of forms built over the space $(\phi^a, c^a, \lambda_a, \bar{c}_a)$ but only of forms over the space \mathcal{M} whose coordinates are (ϕ^a) . Our \mathcal{H} is the Lie-derivative for this last space. Via our variables it is also possible to build vector and multivector fields over \mathcal{M} and to reproduce the full Cartan calculus. For details we refer the reader to ref. [6] and for a deeper geometrical understanding of our enlarged space $(\phi^a, c^a, \lambda_a, \bar{c}_a)$ we invite the interested reader to consult refs. [10].

The reader may remember that the concept of Lie-derivative was mentioned also in the second of refs. [8]. There anyhow the connection between Lie-derivative and Hamiltonian was not as direct as here. Moreover the Lie-derivative was not associated to the flow associated to the conformal potential but with the flow associated to the superpotential (24).

The Hamiltonian $\widetilde{\mathcal{H}}$ has various *universal* symmetries [6] all of which have been studied geometrically. The associated charges² are:

TABLE 3

Q_{BRS}	$=$	$ic^a \lambda_a$
\bar{Q}_{BRS}	$=$	$i\bar{c}_a \omega^{ab} \lambda_b$
Q_g	$=$	$c^a \bar{c}_a$
C	$=$	$\frac{\omega_{ab} c^a c^b}{2}$
\bar{C}	$=$	$\frac{\omega^{ab} \bar{c}_a \bar{c}_b}{2}$
N_H	$=$	$c^a \partial_a H$
\bar{N}_H	$=$	$\bar{c}_a \omega^{ab} \partial_b H$

Using the correspondence between Grassmannian variables and forms, the Q_{BRS} turns out to be nothing else [6] than the exterior derivative³ on phase space and, as it is well known [7] it always commutes with any Lie-derivative. The Q_g , or ghost charge, is the form-number which is always conserved by the Lie-derivative. Similar geometrical meanings can be found [6] for the other charges that are listed above. Of course linear

²The charges which here have been indicated as C and \bar{C} are those that in ref. [6] were called K and \bar{K} . We change their names in order not to confuse them with the K operator of the conformal algebra (see **TABLE 1**).

³We denoted some charges as BRS and anti-BRS charges (Q_{BRS}, \bar{Q}_{BRS}) because they are exterior derivatives as the gauge-BRS charges are and because they are nilpotent and anticommutes among themselves. The ω_{ab} which appears in this table is the inverse of the ω^{ab} of eq. (25).

combinations of them are also conserved and there are two combinations which deserve our attention. They are the following charges:

$$Q_H \equiv Q_{BRS} - \beta N_H; \quad \overline{Q}_H \equiv \overline{Q}_{BRS} + \beta \overline{N}_H; \quad (40)$$

(where β is a dimensionful parameter) which are true supersymmetry charges because, besides commuting with $\widetilde{\mathcal{H}}$, they give⁴:

$$[Q_H, \overline{Q}_H] = 2i\beta\widetilde{\mathcal{H}}. \quad (41)$$

This proves that our $\widetilde{\mathcal{H}}$ is supersymmetric. To be precise it is an $N = 2$ supersymmetry. One realizes immediately that H acts as a sort of superpotential for the supersymmetric Hamiltonian $\widetilde{\mathcal{H}}$. All this basically means that we can obtain a supersymmetric Hamiltonian $\widetilde{\mathcal{H}}$ out of any system with Hamiltonian H and, besides, our $\widetilde{\mathcal{H}}$ has a nice geometrical meaning being the Lie-derivative of the Hamiltonian flow generated by H .

We will now build the $\widetilde{\mathcal{H}}$ of the conformal invariant system given by the Hamiltonian of eq.(8), that means we insert the H of eq.(8) into the $\widetilde{\mathcal{H}}$ of eq.(34). The result is:

$$\widetilde{\mathcal{H}} = \lambda_q p + \lambda_p \frac{g}{q^3} + i\bar{c}_q c^p - 3i\bar{c}_p c^q \frac{g}{q^4} \quad (42)$$

where the indices $(.)^q$ and $(.)^p$ on the variables (λ, c, \bar{c}) replace the indices $(.)^a$ which appeared in the general formalism. In fact, as the system is one-dimensional, the index “ $(.)^a$ ” can only indicate the variables “ (p, q) ” and that is why we use (p, q) as index. The two supersymmetric charges of eq.(40) are in this case

$$Q_H = Q_{BRS} + \beta \left(\frac{g}{q^3} c^q - p c^p \right) \quad (43)$$

$$\overline{Q}_H = \overline{Q}_{BRS} + \beta \left(\frac{g}{q^3} \bar{c}_p + p c_q \right). \quad (44)$$

It was one of the central points of the original paper [1] on conformal mechanics that the Hamiltonians of the system could be, beside H_0 of eq.(16), also D_0 or K_0 of eqs. (17)(18) or any linear combination of them. In the same manner as we built the Lie-derivative $\widetilde{\mathcal{H}}$ associated to H_0 , we can also build the Lie-derivatives associated to the flow generated by D_0 and K_0 . We just have to insert⁵ D_0 or K_0 in place of H as superpotential in the $\widetilde{\mathcal{H}}$ of eq.(34). Calling the associated Lie-derivatives as $\widetilde{\mathcal{D}}_0$ and $\widetilde{\mathcal{K}}_0$, what we get is:

⁴The commutators used below are those of our path-integral for classical mechanics derived in eq.(37).

⁵We will neglect ordering problem in the expression (17) of D_0 because we are doing a classical theory. The sub-index “ $(.)_0$ ” that we will put on $\widetilde{\mathcal{D}}$ and $\widetilde{\mathcal{K}}$ below is to indicate that they were built from the D_0 and K_0 .

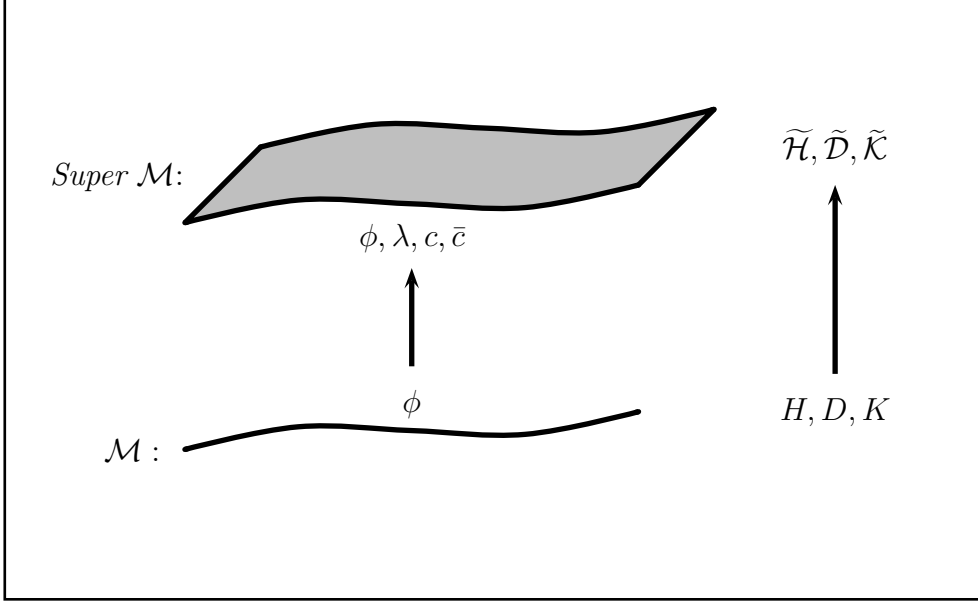


Figure 1: The correspondence between \mathcal{M} and $Super\mathcal{M}$.

$$\tilde{\mathcal{D}}_0 = \frac{1}{2}[\lambda_p p - \lambda_q q + i(\bar{c}_p c^p - \bar{c}_q c^q)] \quad (45)$$

$$\tilde{\mathcal{K}}_0 = -\lambda_p q - i\bar{c}_p c^q \quad (46)$$

The construction is best illustrated in Figure 1.

As it is easy to prove, both $\tilde{\mathcal{D}}_0$ and $\tilde{\mathcal{K}}_0$ are supersymmetric. It is possible in fact to introduce the following charges:

$$Q_D = Q_{BRS} + \gamma(qc^p + pc^q); \quad (47)$$

$$\bar{Q}_D = \bar{Q}_{BRS} + \gamma(p\bar{c}_p - q\bar{c}_q); \quad (48)$$

$$Q_K = Q_{BRS} - \alpha qc^q; \quad (49)$$

$$\bar{Q}_K = \bar{Q}_{BRS} - \alpha q\bar{c}_p; \quad (50)$$

(γ and α play the same role as β for H) which close on $\tilde{\mathcal{D}}_0$ and $\tilde{\mathcal{K}}_0$:

$$[Q_D, \bar{Q}_D] = 4i\gamma\tilde{\mathcal{D}}_0 \quad (51)$$

$$[Q_K, \bar{Q}_K] = 2i\alpha\tilde{\mathcal{K}}_0. \quad (52)$$

One further point to notice is that the conformal algebra of eqs.(11)(12)(13) is now realized, via the commutators (37) of our formalism, by the $(\widetilde{\mathcal{H}}, \widetilde{\mathcal{D}}_0, \widetilde{\mathcal{K}}_0)$ and not by the old functions (H, D_0, K_0) . In fact, via these new commutators, we get:

$$[\widetilde{\mathcal{H}}, \widetilde{\mathcal{D}}_0] = i\widetilde{\mathcal{H}}; \quad [\widetilde{\mathcal{K}}_0, \widetilde{\mathcal{D}}_0] = -i\widetilde{\mathcal{K}}_0; \quad [\widetilde{\mathcal{H}}, \widetilde{\mathcal{K}}_0] = 2i\widetilde{\mathcal{D}}_0; \quad (53)$$

$$[H, D_0] = 0; \quad [K_0, D_0] = 0; \quad [H, K_0] = 0. \quad (54)$$

The next thing to find out, assuming as basic Hamiltonian the $\widetilde{\mathcal{H}}$ and as supersymmetries the ones generated by Q_H and \overline{Q}_H , is to perform the commutators between supersymmetries and conformal operators so to get the superconformal generators. It is easy to work this out and we get:

$$[Q_H, \widetilde{\mathcal{D}}_0] = i(Q_H - Q_{BRS}); \quad [\overline{Q}_H, \widetilde{\mathcal{D}}_0] = i(\overline{Q}_H - \overline{Q}_{BRS}); \quad (55)$$

$$[Q_H, \widetilde{\mathcal{K}}_0] = \frac{i\beta}{\gamma}(Q_D - Q_{BRS}); \quad [\overline{Q}_H, \widetilde{\mathcal{K}}_0] = \frac{i\beta}{\gamma}(\overline{Q}_D - \overline{Q}_{BRS}). \quad (56)$$

From what we have above we realize immediately the role of the Q_D and \overline{Q}_D : besides being the “square roots” of $\widetilde{\mathcal{D}}_0$ they are also (combined with the Q_{BRS} and \overline{Q}_{BRS}) the generators of the superconformal transformations. It is also a simple calculation to evaluate the commutators between the various “supercharges” $Q_H, \overline{Q}_D, \overline{Q}_K, \overline{Q}_H, Q_D, Q_K$:

$$[Q_H, \overline{Q}_D] = i\beta\widetilde{\mathcal{H}} + 2i\gamma\widetilde{\mathcal{D}}_0 - 2\beta\gamma H; \quad [\overline{Q}_H, Q_D] = i\beta\widetilde{\mathcal{H}} + 2i\gamma\widetilde{\mathcal{D}}_0 + 2\beta\gamma H; \quad (57)$$

$$[Q_K, \overline{Q}_D] = i\alpha\widetilde{\mathcal{K}}_0 + 2i\gamma\widetilde{\mathcal{D}}_0 + 2\alpha\gamma K_0; \quad [\overline{Q}_K, Q_D] = i\alpha\widetilde{\mathcal{K}}_0 + 2i\gamma\widetilde{\mathcal{D}}_0 - 2\alpha\gamma K_0; \quad (58)$$

$$[Q_H, \overline{Q}_K] = i\beta\widetilde{\mathcal{H}} + i\alpha\widetilde{\mathcal{K}}_0 - 2\alpha\beta D_0; \quad [\overline{Q}_H, Q_K] = i\beta\widetilde{\mathcal{H}} + i\alpha\widetilde{\mathcal{K}}_0 + 2\alpha\beta D_0; \quad (59)$$

From the RHS of these expressions we see that one needs also the old functions (H, D_0, K_0) in order to close the algebra.

The complete set of operators which close the algebra is listed in the following table:

TABLE 4

$$\begin{aligned}
\widetilde{\mathcal{H}} &= \lambda_q p + \lambda_p \frac{g}{q^3} + i\bar{c}_q c^p - 3i\bar{c}_p c^q \frac{g}{q^4}; & H &= \frac{1}{2} \left(p^2 + \frac{g}{q^2} \right); \\
\widetilde{\mathcal{K}}_0 &= -\lambda_p q - i\bar{c}_p c^q; & K_0 &= \frac{1}{2} q^2; \\
\widetilde{\mathcal{D}}_0 &= \frac{1}{2} [\lambda_p p - \lambda_q q + i(\bar{c}_p c^p - \bar{c}_q c^q)]; & D_0 &= -\frac{1}{2} qp; \\
Q_{BRS} &= i(\lambda_q c^q + \lambda_p c^p); & \overline{Q}_{BRS} &= i(\lambda_p \bar{c}_q - \lambda_q \bar{c}_p); \\
Q_H &= Q_{BRS} + \beta \left(\frac{g}{q^3} c^q - p c^p \right); & \overline{Q}_H &= \overline{Q}_{BRS} + \beta \left(\frac{g}{q^3} \bar{c}_p + p \bar{c}_q \right); \\
Q_K &= Q_{BRS} - \alpha q c^q; & \overline{Q}_K &= \overline{Q}_{BRS} - \alpha q \bar{c}_p; \\
Q_D &= Q_{BRS} + \gamma (q c^p + p c^q); & \overline{Q}_D &= \overline{Q}_{BRS} + \gamma (p \bar{c}_p - q \bar{c}_q);
\end{aligned}$$

The complete algebra among these generators is:

TABLE 5

$$\begin{aligned}
[\widetilde{\mathcal{H}}, \widetilde{\mathcal{D}}_0] &= i\widetilde{\mathcal{H}}; & [\widetilde{\mathcal{K}}, \widetilde{\mathcal{D}}_0] &= -i\widetilde{\mathcal{K}}_0; & [\widetilde{\mathcal{H}}, \widetilde{\mathcal{K}}_0] &= 2i\widetilde{\mathcal{D}}_0; \\
[Q_H, \widetilde{\mathcal{H}}] &= 0; & [\overline{Q}_H, \widetilde{\mathcal{H}}] &= 0; & [Q_H, \overline{Q}_H] &= 2i\beta\widetilde{\mathcal{H}}; \\
[Q_H, \widetilde{\mathcal{D}}_0] &= i(Q_H - Q_{BRS}); & [\overline{Q}_H, \widetilde{\mathcal{D}}_0] &= i(\overline{Q}_H - \overline{Q}_{BRS}); & [Q_{BRS}, \widetilde{\mathcal{D}}] &= [\overline{Q}_{BRS}, \widetilde{\mathcal{D}}] = 0; \\
[Q_H, \widetilde{\mathcal{K}}_0] &= i\beta\gamma^{-1}(Q_D - Q_{BRS}); & [\overline{Q}_H, \widetilde{\mathcal{K}}_0] &= i\beta\gamma^{-1}(\overline{Q}_D - \overline{Q}_{BRS}); & & \\
[Q_{BRS}, \widetilde{\mathcal{H}}] &= [\overline{Q}_{BRS}, \widetilde{\mathcal{H}}] = 0; & [Q_{BRS}, \widetilde{\mathcal{K}}] &= [\overline{Q}_{BRS}, \widetilde{\mathcal{K}}] = 0; & & \\
[Q_D, \widetilde{\mathcal{H}}] &= -2i\gamma\beta^{-1}(Q_H - Q_{BRS}); & [\overline{Q}_D, \widetilde{\mathcal{H}}] &= -2i\gamma\beta^{-1}(\overline{Q}_H - \overline{Q}_{BRS}); & & \\
[Q_D, \widetilde{\mathcal{K}}_0] &= 2i\gamma\alpha^{-1}(Q_K - Q_{BRS}); & [\overline{Q}_D, \widetilde{\mathcal{K}}_0] &= 2i\gamma\alpha^{-1}(\overline{Q}_K - \overline{Q}_{BRS}); & [Q_D, \overline{Q}_D] &= 4i\gamma\widetilde{\mathcal{D}}_0; \\
[Q_D, \widetilde{\mathcal{D}}_0] &= 0; & [\overline{Q}_D, \widetilde{\mathcal{D}}_0] &= 0; & & \\
[Q_K, \widetilde{\mathcal{H}}] &= -i\alpha\gamma^{-1}(Q_D - Q_{BRS}); & [\overline{Q}_K, \widetilde{\mathcal{H}}] &= -i\alpha\gamma^{-1}(\overline{Q}_D - \overline{Q}_{BRS}); & & \\
[Q_K, \widetilde{\mathcal{D}}_0] &= -i(Q_K - Q_{BRS}); & [\overline{Q}_K, \widetilde{\mathcal{D}}_0] &= -i(\overline{Q}_K - \overline{Q}_{BRS}); & [Q_K, \overline{Q}_K] &= 2i\alpha\widetilde{\mathcal{K}}_0; \\
[Q_K, \widetilde{\mathcal{K}}_0] &= 0; & [\overline{Q}_K, \widetilde{\mathcal{K}}_0] &= 0; & & \\
[Q_H, \overline{Q}_D] &= i\beta\widetilde{\mathcal{H}} + 2i\gamma\widetilde{\mathcal{D}}_0 - 2\beta\gamma H; & [\overline{Q}_H, Q_D] &= i\beta\widetilde{\mathcal{H}} + 2i\gamma\widetilde{\mathcal{D}}_0 + 2\beta\gamma H; & & \\
[Q_K, \overline{Q}_D] &= i\alpha\widetilde{\mathcal{K}}_0 + 2i\gamma\widetilde{\mathcal{D}}_0 + 2\alpha\gamma K; & [\overline{Q}_K, Q_D] &= i\alpha\widetilde{\mathcal{K}}_0 + 2i\gamma\widetilde{\mathcal{D}}_0 - 2\alpha\gamma K; & & \\
[Q_H, \overline{Q}_K] &= i\beta\widetilde{\mathcal{H}} + i\alpha\widetilde{\mathcal{K}}_0 - 2\alpha\beta D; & [\overline{Q}_H, Q_K] &= i\beta\widetilde{\mathcal{H}} + i\alpha\widetilde{\mathcal{K}}_0 + 2\alpha\beta D; & & \\
[Q_H, \overline{Q}_{BRS}] &= [\overline{Q}_H, Q_{BRS}] = i\beta\widetilde{\mathcal{H}}; & [Q_K, \overline{Q}_{BRS}] &= [\overline{Q}_K, Q_{BRS}] = i\alpha\widetilde{\mathcal{K}}_0; & & \\
[Q_D, \overline{Q}_{BRS}] &= [\overline{Q}_D, Q_{BRS}] = 2i\gamma\widetilde{\mathcal{D}}_0; & & & & \\
[Q_{(\dots)}, H] &= \beta^{-1}(Q_{BRS} - Q_H); & [\overline{Q}_{(\dots)}, H] &= \beta^{-1}(\overline{Q}_{BRS} - \overline{Q}_H); & & \\
[Q_{(\dots)}, D_0] &= (2\gamma)^{-1}(Q_{BRS} - Q_D); & [\overline{Q}_{(\dots)}, D_0] &= (2\gamma)^{-1}(\overline{Q}_{BRS} - \overline{Q}_D); & & \\
[Q_{(\dots)}, K_0] &= \alpha^{-1}(Q_{BRS} - Q_K); & [\overline{Q}_{(\dots)}, K_0] &= \alpha^{-1}(\overline{Q}_{BRS} - \overline{Q}_K); & & \\
[\widetilde{\mathcal{H}}, H] &= 0; & [\widetilde{\mathcal{K}}_0, K_0] &= 0; & [\widetilde{\mathcal{D}}_0, D_0] &= 0; \\
[\widetilde{\mathcal{H}}, K_0] &= [H, \widetilde{\mathcal{K}}_0] = 2iD; & [\widetilde{\mathcal{H}}, D_0] &= [H, \widetilde{\mathcal{D}}_0] = iH; & [\widetilde{\mathcal{D}}_0, K_0] &= [D_0, \widetilde{\mathcal{K}}_0] = iK.
\end{aligned}$$

All other commutators⁶ are zero.

We notice than for our supersymmetric extension we need 14 charges (see **TABLE 4**) in order for the algebra to close, while in the extension of ref. [4] one needs only 8 charges (see **TABLE 1**). This is so not only because ours is an $N = 2$ supersymmetry (while the one of [4] is an $N = 1$) but also because of the totally different character of the model.

4 Study of the two Superconformal Algebras

A Lie superalgebra [12] is an algebra made of even E_n and odd O_α generators whose graded commutators look like:

$$[E_m, E_n] = F_{mn}^p E_p; \quad (60)$$

$$[E_m, O_\alpha] = G_{m\alpha}^\beta O_\beta; \quad (61)$$

$$[O_\alpha, O_\beta] = C_{\alpha\beta}^m E_m; \quad (62)$$

and where the structure constants $F_{mn}^p, G_{m,\alpha}^\beta, C_{\alpha,\beta}^m$ satisfy generalized Jacobi identities.

One can interpret the relation (61) as saying that the even part of the algebra has a representation on the odd part. This is clear if we consider the odd part as a vector space and that the even part acts on this vector space via the graded commutators. The structure constants $F_{m\alpha}^\beta$ are then the matrix elements which characterize the representations.

For superconformal algebras the usual folklore says that the even part of the algebra has his conformal subalgebra represented spinorially on the odd part. The reasoning roughly goes as follows: the odd part of the algebra must contain the supersymmetry generators which transform as spinors under the Lorentz group which is a subgroup of the conformal algebra. So it is impossible that the whole conformal algebra is represented non-spinorially on the odd part.

Actually this line of reasoning is true in a relativistic context in which the supersymmetry is a true relativistic supersymmetry and the charges must carry a spinor index due to their nature. In our non-relativistic point particle case instead the charges do not carry any space-time index and so we do not have as a consequence that necessarily the even part of the algebra is represented spinorially on the odd part. It can happen but it can also not happen. In this respect we will analyze the superalgebras of the two supersymmetric extensions of conformal mechanics seen here, the one of [4] and ours.

⁶The $Q_{(\dots)}$ appearing in the table can be any of the following operators: Q_{BRS}, Q_H, Q_D, Q_K and the same holds for $\overline{Q}_{(\dots)}$. Obviously all commutators are between quantities calculated at the same time.

Let us start from the one of ref. [4] which is given in **TABLE 1**. The conformal subalgebra \mathcal{G}_0 of the even part can be organized in an $SO(2, 1)$ form as follows:

$$\mathcal{G}_0 : \begin{cases} B_1 = \frac{1}{2} \left[\frac{K}{a} - aH \right] \\ B_2 = D \\ J_3 = \frac{1}{2} \left[\frac{K}{a} + aH \right] \end{cases}$$

where a is the same parameter introduced in [1] with dimension of time.

The odd part \mathcal{G}_1 is:

$$\mathcal{G}_1 : \begin{cases} Q \\ Q^\dagger \\ S \\ S^\dagger \end{cases}$$

It is easy to work out, using the results of **TABLE 2**, the action of the \mathcal{G}_0 on \mathcal{G}_1 . The result is summarized in the following table:

TABLE 6

$[B_1, Q] = \frac{1}{2a}S;$	$[B_2, Q] = -\frac{i}{2}Q;$	$[J_3, Q] = \frac{1}{2a}S;$
$[B_1, Q^\dagger] = -\frac{1}{2a}S^\dagger;$	$[B_2, Q^\dagger] = -\frac{i}{2}Q^\dagger;$	$[J_3, Q^\dagger] = -\frac{1}{2a}S^\dagger;$
$[B_1, S] = -\frac{a}{2}Q;$	$[B_2, S] = \frac{i}{2}S;$	$[J_3, S] = \frac{a}{2}Q;$
$[B_1, S^\dagger] = \frac{a}{2}Q^\dagger;$	$[B_2, S^\dagger] = \frac{i}{2}S^\dagger;$	$[J_3, S^\dagger] = -\frac{a}{2}Q^\dagger;$

As we said before, in order to act with the even part of the algebra on the odd part, we have to consider the odd part of the as a vector space. Let us then introduce the following “vectors”:

$$|q\rangle \equiv Q + Q^\dagger \tag{63}$$

$$|p\rangle \equiv S - S^\dagger \tag{64}$$

$$|r\rangle \equiv Q - Q^\dagger \tag{65}$$

$$|s\rangle \equiv S + S^\dagger; \tag{66}$$

they label a 4-dimensional vector space. On these vectors we act via the commutators, for example:

$$B_1|q\rangle \equiv [B_1, Q + Q^\dagger] \quad (67)$$

It is then immediate to realize from **TABLE 6** that the 2-dim. space with basis $(|q\rangle, |p\rangle)$ form a closed space under the action of even part of the algebra so it carries a 2-dim. representation and the same holds for the space $(|r\rangle, |s\rangle)$. We can immediately check which kind of representation is this: Let us take the Casimir operator of the algebra \mathcal{G}_0 which is $\mathcal{C} = B_1^2 + B_2^2 - J_3^2$ and apply it to a state of one of the two 2-dim. representations:

$$\begin{aligned} \mathcal{C}|q\rangle &= [B_1, [B_1, Q + Q^\dagger]] + [B_2, [B_2, Q + Q^\dagger]] - [J_3, [J_3, Q + Q^\dagger]] \\ &= -\frac{3}{4}(Q + Q^\dagger) \\ &= -\frac{3}{4}|q\rangle \end{aligned} \quad (68)$$

This factor $\frac{3}{4} = -\frac{1}{2}(\frac{1}{2} + 1)$ indicates that the $(|q\rangle, |p\rangle)$ space carries a spinorial representation. It is possible to prove the same for the other space.

Let us now turn the same crank for our supersymmetric extension of conformal mechanics. Looking at the **TABLE 4** of our operators, we can organize the even part \mathcal{G}_0 , as follows:

TABLE 7 (\mathcal{G}_0)

$\begin{aligned} B_1 &= \frac{1}{2} \left(\frac{\tilde{\mathcal{K}}}{a} - a\tilde{\mathcal{H}} \right); & P_1 &= 2D; \\ B_2 &= \tilde{\mathcal{D}}; & P_2 &= aH - \frac{K}{a}; \\ J_3 &= \frac{1}{2} \left(\frac{\tilde{\mathcal{K}}}{a} + a\tilde{\mathcal{H}} \right); & P_0 &= aH + \frac{K}{a}; \end{aligned}$
--

The LHS is the usual $SO(2, 1)$ while the RHS is formed by three translations because they commute among themselves. So the overall algebra is the Euclidean group $E(2, 1)$.

The odd part of our superalgebra is made of 8 operators (see **TABLE 4**) which are:

TABLE 8 (\mathcal{G}_1)

$Q_H;$	$\overline{Q}_H;$
$Q_K;$	$\overline{Q}_K;$
$Q_D;$	$\overline{Q}_D;$
$Q_{BRS};$	$\overline{Q}_{BRS};$

As we did before in **TABLE 6** for the model of [4], we will now evaluate for our model the action of \mathcal{G}_0 on \mathcal{G}_1 . The result is summarized in the next table:

TABLE 9

$[B_1, Q_H] = \frac{i}{2\eta}(Q_{BRS} - Q_D);$	$[B_1, \overline{Q}_H] = \frac{i}{2\eta}(\overline{Q}_{BRS} - \overline{Q}_D);$
$[B_1, Q_K] = \frac{i}{2\eta}(Q_{BRS} - Q_D);$	$[B_1, \overline{Q}_K] = \frac{i}{2\eta}(\overline{Q}_{BRS} - \overline{Q}_D);$
$[B_1, Q_D] = -i\eta(Q_H + Q_K - 2Q_{BRS});$	$[B_1, \overline{Q}_D] = -i\eta(\overline{Q}_H + \overline{Q}_K - 2\overline{Q}_{BRS});$
$[B_1, Q_{BRS}] = 0;$	$[B_1, \overline{Q}_{BRS}] = 0;$
$[B_2, Q_H] = i(Q_{BRS} - Q_H);$	$[B_2, \overline{Q}_H] = i(\overline{Q}_{BRS} - \overline{Q}_H);$
$[B_2, Q_K] = i(Q_K - Q_{BRS});$	$[B_2, \overline{Q}_K] = i(\overline{Q}_K - \overline{Q}_{BRS});$
$[B_2, Q_D] = 0;$	$[B_2, \overline{Q}_D] = 0;$
$[B_2, Q_{BRS}] = 0;$	$[B_2, \overline{Q}_{BRS}] = 0;$
$[J_3, Q_H] = \frac{i}{2\eta}(Q_{BRS} - Q_D);$	$[J_3, \overline{Q}_H] = \frac{i}{2\eta}(\overline{Q}_{BRS} - \overline{Q}_D);$
$[J_3, Q_K] = -\frac{i}{2\eta}(Q_{BRS} - Q_D);$	$[J_3, \overline{Q}_K] = -\frac{i}{2\eta}(\overline{Q}_{BRS} - \overline{Q}_D);$
$[J_3, Q_D] = i\eta(Q_H - Q_K);$	$[J_3, \overline{Q}_H] = i\eta(\overline{Q}_H - \overline{Q}_K);$
$[J_3, Q_{BRS}] = 0;$	$[J_3, \overline{Q}_{BRS}] = 0;$
$[P_1, Q_{(\dots)}] = \gamma^{-1}(Q_D - Q_{BRS});$	$[P_1, \overline{Q}_{(\dots)}] = -\gamma^{-1}(\overline{Q}_D - \overline{Q}_{BRS});$
$[P_2, Q_{(\dots)}] = \gamma^{-1}\eta(Q_H - Q_K);$	$[P_2, \overline{Q}_{(\dots)}] = -\gamma^{-1}\eta(\overline{Q}_H - \overline{Q}_K);$
$[P_0, Q_{(\dots)}] = \gamma^{-1}\eta(Q_H + Q_K - 2Q_{BRS});$	$[P_0, \overline{Q}_{(\dots)}] = -\gamma^{-1}\eta(\overline{Q}_H + \overline{Q}_K - 2\overline{Q}_{BRS});$

where for simplicity we have made the choice $a = \sqrt{\frac{\beta}{\alpha}}$ and $\eta \equiv \frac{\gamma}{\sqrt{\alpha\beta}}$.

As we have to represent the conformal subalgebra of \mathcal{G}_0 (see **TABLE 7**) on the vector space \mathcal{G}_1 of **TABLE 8** it is easy to realize from **TABLE 9** that the following three vectors

$$\begin{cases} |q_H\rangle = (Q_H - Q_{BRS}) - (\overline{Q}_H - \overline{Q}_{BRS}) \\ |q_K\rangle = (Q_K - Q_{BRS}) - (\overline{Q}_K - \overline{Q}_{BRS}) \\ |q_D\rangle = \eta^{-1}[(Q_D - Q_{BRS}) - (\overline{Q}_D - \overline{Q}_{BRS})] \end{cases} \quad (69)$$

make an irreducible representation of the conformal subalgebra. In fact, using **TABLE 9**, we get:

$$\begin{cases} B_1|q_H\rangle = -\frac{i}{2}|q_D\rangle \\ B_2|q_H\rangle = -i|q_H\rangle \\ J_3|q_H\rangle = -\frac{i}{2}|q_D\rangle \\ B_1|q_K\rangle = -\frac{i}{2}|q_D\rangle \\ B_2|q_K\rangle = i|q_K\rangle \\ J_3|q_K\rangle = \frac{i}{2}|q_D\rangle \\ B_1|q_D\rangle = -i(|q_H\rangle + |q_K\rangle) \\ B_2|q_D\rangle = 0 \\ J_3|q_D\rangle = i(|q_H\rangle - |q_K\rangle). \end{cases} \quad (70)$$

Having three vectors in this representation we presume it is an ‘integer’ spin representation, but to be sure let us apply the Casimir operator on a vector. The Casimir is given, as before, by: $\mathcal{C} = B_1^2 + B_2^2 - J_3^2$ but we must remember to use as B_1 , B_2 and J_3 the operators contained in **TABLE 7**. It is then easy to check that

$$\mathcal{C}|q_H\rangle = -2|q_K\rangle. \quad (71)$$

The same we get for the other two vectors $|q_K\rangle, |q_D\rangle$, so the eigenvalue in the equation above is $-2 = -1(1 + 1)$ and this indicates that those vectors make a “spin” 1 representation.

In the same way as before it is easy to prove that these other three vectors:

$$\begin{cases} |\widetilde{q}_H\rangle = (Q_H - Q_{BRS}) + (\overline{Q}_H - \overline{Q}_{BRS}) \\ |\widetilde{q}_K\rangle = (Q_K - Q_{BRS}) + (\overline{Q}_K - \overline{Q}_{BRS}) \\ |\widetilde{q}_D\rangle = (Q_D - Q_{BRS}) + (\overline{Q}_D - \overline{Q}_{BRS}) \end{cases} \quad (72)$$

make another irreducible representation of “spin” 1.

Of course, as the vector space \mathcal{G}_1 of **TABLE 8** is 8-dimensional and up to now we have used only 6 vectors to build the two integer representations, we expect that there must be some other representations which can be built using the two remaining vectors. It is in fact so. We can build the following two other vectors:

$$|q_{BRS}\rangle = Q_{BRS} - \overline{Q}_{BRS} \quad (73)$$

$$|\widetilde{q}_{BRS}\rangle = Q_{BRS} + \overline{Q}_{BRS} \quad (74)$$

and it is easy to see that each of them carry a representation of spin zero:

$$\mathcal{C}|q_{BRS}\rangle = \mathcal{C}|\widetilde{q}_{BRS}\rangle = 0 \quad (75)$$

So we can conclude that our vector space \mathcal{G}_1 carries a reducible representation of the conformal algebra made of two spin one and two spin zero representations.

We wanted to do this analysis in order to underline a further difference between our supersymmetric extension and the one of [4] whose odd part \mathcal{G}_1 , as we showed before, carries two spin one-half representations.

One last thing to do is to find out to which of the superalgebras classified in the literature ours belongs. We will come back to this in the future.

5 Superspace Formulation of the Model

It is easy and instructive to do a superspace formulation of our model like the authors of ref.[4] did for theirs.

Let us enlarge our “base space” (t) to a superspace $(t, \theta, \bar{\theta})$ where $(\theta, \bar{\theta})$ are Grassmannian partners of (t) . It is then possible to put all the variables $(\phi^a, c^a, \lambda_a, \bar{c}_a)$ in a single superfield Φ defined as follows:

$$\Phi^a(t, \theta, \bar{\theta}) = \phi^a(t) + \theta c^a(t) + \bar{\theta} \omega^{ab} \bar{c}_b(t) + i\theta\bar{\theta} \omega^{ab} \lambda_b(t) \quad (76)$$

This superfield had already been introduced in ref.[6]. It is a scalar field under the supersymmetry transformations of the system. The various factors of “ i ” appearing in its definition are due to the fact that we chose [6] the c^a, \bar{c}_a to be real and the $\theta, \bar{\theta}$ to be pure imaginary.

It is a simple exercise to find the expansion of any function $F(\Phi^a)$ of the superfields in terms of $\theta, \bar{\theta}$. For example, choosing as function the Hamiltonian H of a system, we get:

$$H(\Phi^a) = H(\phi) + \theta N_H - \bar{\theta} \bar{N}_H + i\theta\bar{\theta} \widetilde{\mathcal{H}} \quad (77)$$

where N_H and \bar{N}_H and $\widetilde{\mathcal{H}}$ are those given in **TABLE 3** and in eq.(34).

From eq.(77) it is easy to prove that:

$$i \int H(\Phi) d\theta d\bar{\theta} = \widetilde{\mathcal{H}} \quad (78)$$

Here we immediately notice a crucial difference with the supersymmetric QM model of ref.[4]. In the language of superfields (see the second of ref. [4]) those authors obtain the supersymmetric potential of their Hamiltonian by inserting the superfield into the superpotential (which is given by eq.(24)) and integrating in something like $\theta, \bar{\theta}$, while we get the potential part of our supersymmetric Hamiltonian by inserting the superfield into the normal potential of the conformal mechanical model given in (16).

The space $(\phi^a, c^a, \lambda_a, \bar{c}_a)$ somehow can be considered as a *target* space whose *base* space is the superspace $(t, \theta, \bar{\theta})$. The action of the various charges listed in our **TABLE 4** is on the target-space variables but we can consider it as induced by some transformations on the base-space. If we collectively indicate the charges acting on $(\phi^a, c^a, \lambda_a, \bar{c}_a)$ as Ω , we shall indicate the generators of the corresponding transformations on the base space as $\widehat{\Omega}$. The relation between the two is the following:

$$\delta\Phi^a = -\varepsilon\widehat{\Omega}\Phi^a \quad (79)$$

where

$$\delta\Phi^a = [\varepsilon\Omega, \Phi^a] \quad (80)$$

with ε the commuting or anticommuting infinitesimal parameter of our transformations⁷ and $[(\cdot), (\cdot)]$ the graded commutators of eq.(37).

Using the relations above it is easy to work out the superspace representation of the operators contained in **TABLE 3**, they are given in the table below:

TABLE 10

\widehat{Q}_{BRS}	$=$	$-\partial_\theta$
$\widehat{\bar{Q}}_{BRS}$	$=$	$\partial_{\bar{\theta}}$
\widehat{Q}_g	$=$	$\bar{\theta}\partial_{\bar{\theta}} - \theta\partial_\theta$
$\widehat{\bar{C}}$	$=$	$\bar{\theta}\partial_\theta$
$\widehat{\bar{C}}$	$=$	$\theta\partial_{\bar{\theta}}$
\widehat{N}_H	$=$	$\bar{\theta}\partial_t$
$\widehat{\bar{N}}_H$	$=$	$\theta\partial_t$

Via the charges above it is immediate to write down also the supersymmetric charges of eq.(40):

$$\widehat{Q}_H = -\partial_\theta - \beta \bar{\theta}\partial_t; \quad \widehat{\bar{Q}}_H = \partial_{\bar{\theta}} + \beta \theta\partial_t; \quad (81)$$

Their anticommutator gives:

$$[\widehat{Q}_H, \widehat{\bar{Q}}_H] = -2\beta \frac{\partial}{\partial t} \quad (82)$$

⁷The conventions (79) and (80) are slightly different than the ones in ref. [6]. Here we also correct some misprints present in that reference.

from which, comparing the above equation with eq.(41), one gets the superspace representation of $\widehat{\mathcal{H}}$:

$$\widehat{\mathcal{H}} = i \frac{\partial}{\partial t}. \quad (83)$$

Proceeding in the same way, via the relations (79), (80), it is a long but easy procedure to give a superspace representation to the charges $Q_D, \overline{Q}_D, Q_K, \overline{Q}_K$ of eqs. (47)-(50). This long derivation is contained in the appendix and the result is:

$$\widehat{Q}_K = -\frac{\partial}{\partial \theta} - \alpha \omega^{ad} K_{db} \bar{\theta} \quad (84)$$

$$\widehat{\overline{Q}}_K = \frac{\partial}{\partial \bar{\theta}} + \alpha \omega^{ad} K_{db} \theta \quad (85)$$

$$\widehat{Q}_D = -\frac{\partial}{\partial \theta} - 2\gamma \omega^{ad} D_{db} \bar{\theta} \quad (86)$$

$$\widehat{\overline{Q}}_D = \frac{\partial}{\partial \bar{\theta}} + 2\gamma \omega^{ad} D_{db} \theta \quad (87)$$

where the matrices K_{db} and D_{db} are:

$$K_{db} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; \quad D_{db} = -\frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (88)$$

(the repeated indices in eqs.(84)-(87) are summed). The matrices K_{db} and D_{db} are 2×2 just because the symplectic matrix itself ω^{ab} is 2×2 in our case. The conformal mechanics system in fact has just a pair of phase-space variables (p, q) and the index in the ϕ^a -phase space variables can take only 2 values to indicate either q or p (see the eqs. of motion (25)).

From the expressions of $\widehat{Q}_K, \widehat{\overline{Q}}_K, \widehat{Q}_D, \widehat{\overline{Q}}_D$ above we see that they have two free indices. This implies (see eq.(79)) that those operators “turn” the various superfields in the sense that they turn a Φ^q into combinations of Φ^q and Φ^p and viceversa. This is something the other charges did not do.

Reached this point, we should stop and think a little bit about this superspace representation. We gave the superspace representation of the various charges $(Q_D, \overline{Q}_D, Q_K, \overline{Q}_K)$ of eqs.(47)-(50) which were linked to the $\tilde{\mathcal{D}}_0, \tilde{\mathcal{K}}_0$ of eqs.(45)(46). But these last quantities were built using the D_0 and K_0 that is the D and K at $t = 0$. If we had used, in building the $\tilde{\mathcal{D}}_0, \tilde{\mathcal{K}}_0$, the D and K at $t \neq 0$ of eqs.(9) and (10), we would have obtained a $\tilde{\mathcal{D}}$ and a $\tilde{\mathcal{K}}$ different from those of eqs.(45)(46) and which would have had an explicit dependence on t . Consequently also the associated supersymmetric charges $(Q_D^t, \overline{Q}_D^t, Q_K^t, \overline{Q}_K^t)$, having extra terms depending on t , would be different from those of eqs.(47)-(50). Being these charges different, also their superspace representations shall be different from those given in eqs.(84)-(87). The difference at the level of superspace is crucial because it involves t which is part of superspace.

Let us then start this over-all process by first building the explicitly t -dependent $\tilde{\mathcal{D}}$ and $\tilde{\mathcal{K}}$ from the following operators D and K :

$$H = H_0 \quad (89)$$

$$D = tH + D_0 \quad (90)$$

$$K = t^2H + 2tD_0 + K_0 \quad (91)$$

from which we get:

$$\tilde{\mathcal{D}} = t\tilde{\mathcal{H}} + \tilde{\mathcal{D}}_0 \quad (92)$$

$$\tilde{\mathcal{K}} = t^2\tilde{\mathcal{H}} + 2t\tilde{\mathcal{D}}_0 + \tilde{\mathcal{K}}_0. \quad (93)$$

It is easy to understand why these relations hold by remembering the manner we got the Lie-derivatives out of the superpotentials. The explicit form of $\tilde{\mathcal{D}}$ in terms of $(\phi^a, c^a, \lambda_a, \bar{c}_a)$ can be obtained from (92) once we insert the $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{D}}_0$ whose explicit form we already had in eqs.(42) and (45). The same for $\tilde{\mathcal{K}}$. Let us now turn to the form of the associated fermionic charges which we will indicate as $(Q_D^t, Q_K^t, \bar{Q}_D^t, \bar{Q}_K^t)$ where the index “ $(.)^t$ ” is to indicate their explicit dependence on t . As it is shown in formula (A.1) of Appendix A, the Q_D and Q_K can be written using the charges N_D and N_K of (A.2) and the Q_{BRS} . As it is only the $N_{(\dots)}$ and not the Q_{BRS} which pull in quantities like D, K which may depend explicitly on time, we should only concentrate on the $N_{(\dots)}$. From their definition (see eq.(A.2)):

$$N_D = c^a \partial_a D; \quad N_K = c^a \partial_a K \quad (94)$$

we see that applying the operator $c^a \partial_a$ on both sides of eqs.(90)(91), we get:

$$N_D^t = tN_H + N_D \quad (95)$$

$$N_K^t = t^2N_H + 2tN_D + N_K. \quad (96)$$

The next step is to write the Q_D^t and Q_K^t . As they are given in formula (A.1), using that equation and (95)(96) above we get:

$$Q_D^t = Q_{BRS} - 2\gamma N_D^t \quad (97)$$

$$Q_K^t = Q_{BRS} - \alpha N_K^t. \quad (98)$$

In a similar manner, via eq.(A.9) and applying the operator $\bar{c}_a \omega^{ab} \partial_b$ to eqs.(90)(91), we get the \bar{Q}_D^t and \bar{Q}_K^t :

$$\bar{Q}_D^t = \bar{Q}_{BRS} + 2\gamma \bar{N}_D^t \quad (99)$$

$$\bar{Q}_K^t = \bar{Q}_{BRS} + \alpha \bar{N}_K^t. \quad (100)$$

We shall not write down explicitly the expressions of $(Q_D^t, Q_K^t, \bar{Q}_D^t, \bar{Q}_K^t)$ in terms of $(\phi^a, c^a, \lambda_a, \bar{c}_a)$ because we have already in eqs.(47)-(50) and (A.2)(A.9) the expressions⁸ of the various charges $(Q_D, Q_K, \bar{Q}_D, \bar{Q}_K, N_D, N_K, \bar{N}_K, \bar{N}_D)$ which make up, according to eqs.(97)-(100), the new time dependent charges. The next step is to obtain the superspace version of $(Q_D^t, Q_K^t, \bar{Q}_D^t, \bar{Q}_K^t)$. Following a procedure identical to the one explained in detailed in the appendix for the time-independent charges it is easy to get them and, via their anticommutators, to derive the superspace version of the $\tilde{\mathcal{D}}$ and $\tilde{\mathcal{K}}$. All these operators are listed in the table below:

TABLE 11

$\begin{aligned} \widehat{\mathcal{H}} &= i \frac{\partial}{\partial t}; \\ \widehat{\mathcal{D}} &= it \frac{\partial}{\partial t} - \frac{i}{2} \sigma_3; \\ \widehat{\mathcal{K}} &= it^2 \frac{\partial}{\partial t} - it \sigma_3 - i \sigma_-; \\ \hat{Q}_D^t &= -\frac{\partial}{\partial \theta} - 2\gamma \bar{\theta} t \frac{\partial}{\partial t} + \gamma \bar{\theta} \sigma_3; \\ \widehat{\bar{Q}}_D^t &= \frac{\partial}{\partial \bar{\theta}} + 2\gamma \theta t \frac{\partial}{\partial t} - \gamma \theta \sigma_3; \\ \hat{Q}_K^t &= -\frac{\partial}{\partial \theta} - \alpha \bar{\theta} t^2 \frac{\partial}{\partial t} + \alpha t \bar{\theta} \sigma_3 + \alpha \bar{\theta} \sigma_-; \\ \widehat{\bar{Q}}_K^t &= \frac{\partial}{\partial \bar{\theta}} + \alpha \theta t^2 \frac{\partial}{\partial t} - \alpha t \theta \sigma_3 - \alpha \theta \sigma_-; \\ \hat{H} &= \bar{\theta} \theta \frac{\partial}{\partial t}; \\ \hat{D} &= \bar{\theta} \theta (t \frac{\partial}{\partial t} - \frac{1}{2} \sigma_3); \\ \hat{K} &= \bar{\theta} \theta (t^2 \frac{\partial}{\partial t} - t \sigma_3 - \sigma_-). \end{aligned}$

In the previous table the σ_3 and σ_- are the Pauli matrices:

⁸Only be careful in using the D_0 and K_0 in eqs.(A.1) and (A.9).

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (101)$$

The reasons for the presence of these two-dimensional matrices has been explained in the paragraph below eq.(88).

The last three operators listed in **TABLE 11** are the superspace version of the old (H, D, K) . To get this representation we used again and again the rules given by eqs.(79)(80). As their representation looks quite unusual, we have reported the details of their derivations in Appendix B.

We want to conclude this section by presenting a new symmetry of our system. A symmetry which is not among those found up-to-now whose charges we have listed in **TABLE 4**. It is associated with the following superspace operator:

$$\hat{Q}_S = \theta \frac{\partial}{\partial \theta} + \bar{\theta} \frac{\partial}{\partial \bar{\theta}}. \quad (102)$$

This operator is very similar to the ghost-charge \hat{Q}_g of **TABLE 10** but it has a crucial sign difference.

Let us apply it on the RHS of eq.(79) and see which variation it induces on the $(\phi^a, c^a, \lambda_a, \bar{c}_a)$:

$$\delta_{Q_S} \Phi^a = -\varepsilon \hat{Q}_S \Phi^a = -\varepsilon (\theta c^a + \bar{\theta} \omega^{ab} \bar{c}_b + 2i\bar{\theta} \theta \omega^{ab} \lambda_b). \quad (103)$$

Comparing the components with the same number of θ and $\bar{\theta}$ on both side of the equation above, we get

$$\delta_{Q_S} \phi^a = 0 \quad (104)$$

$$\delta_{Q_S} c^a = -\varepsilon c^a \quad (105)$$

$$\delta_{Q_S} \bar{c}_a = -\varepsilon \bar{c}_a \quad (106)$$

$$\delta_{Q_S} \lambda_a = -2\varepsilon \lambda_a. \quad (107)$$

It is easy to check how the Lagrangian of our model (see eq.(31)) changes under the variations above:

$$\delta_{Q_S} \tilde{\mathcal{L}} = -2\varepsilon \tilde{\mathcal{L}}. \quad (108)$$

We can conclude that the transformations induced by \hat{Q}_S on the $(\phi^a, c^a, \lambda_a, \bar{c}_a)$ are a symmetry of our system. In fact they just rescale the overall lagrangian so they keep the equations of motion invariant and these qualifies them as symmetry transformations. Of course they are *non-canonical* symmetries because the rescaling of the whole

lagrangian is not a canonical transformation in $(\phi^a, c^a, \lambda_a, \bar{c}_a)$. As it is not canonical we cannot find a canonical generator in $(\phi^a, c^a, \lambda_a, \bar{c}_a)$ associated to \widehat{Q}_S . Note that the above is a symmetry of any $\tilde{\mathcal{L}}$ not necessarily of the conformal model we have analyzed here. We could ask if this was a symmetry also of the susyQM model of Witten [8] or at least of the conformal QM of ref. [4]. The answer is *No!*, the technical reason being that in those QM models the analogs of the λ_a variables enter the lagrangian with a quadratic term while in our $\tilde{\mathcal{L}}$ they enter linearly. There is also an important physical reason why that symmetry was not present in those QM models while it was present in our CM one. The reason is that in QM one cannot rescale the action (as our symmetry does) because there is the \hbar setting a scale for the action, while it can be done in CM where no scale is set. The reader may object that our transformation rescales the Lagrangian but not the action

$$\tilde{S} = \int \tilde{\mathcal{L}} dt \quad (109)$$

because one could compensate the rescaling of the $\tilde{\mathcal{L}}$ with a rescaling of t . That is not so because our Q_S transforms only the Grassmannian partners of time $(\theta, \bar{\theta})$ and not time itself.

We will come back to this symmetry in the future because it seems to be at the heart of the difference between QM and CM.

6 Exact Solution of the Supersymmetric Model

The original conformal mechanical model was solved exactly in eq.(2.35) of reference [1]. The solution is given by the relation:

$$q^2(t) = 2t^2H - 4tD_0 + 2K_0. \quad (110)$$

As (H, D_0, K_0) are constants of motion, once their values are assigned we stick them in eq. (110), and we get a relation between “ q ” (on the LHS of (110)) and “ t ” on the RHS. This is the solution of the equation of motion with “initial conditions” given by the values we assign to the constants of motion (H, D_0, K_0) . The reader may object that we should give only two constant values (corresponding to the initial conditions $(q(0), \dot{q}(0))$) and not three. Actually the three values assigned to (H, D_0, K_0) are not arbitrary because, as it was proven in eq.(2-36) of ref. [1], these three quantities are linked by a constraint:

$$(HK_0 - D_0^2) = \frac{g}{4} \quad (111)$$

where “ g ” is the coupling which entered the original Hamiltonian (see eq.(1) of the present paper). Having one constraint among the three constants of motion brings them down to two.

The proof of the relation (110) above is quite simple. On the RHS, as the (H, D_0, K_0) are constants of motion, we can replace them with their time dependent expression (H, D, K) (see eq.(89)-(91)), which are explicitly:

$$H = \frac{1}{2} \left(\dot{q}^2(t) + \frac{g}{q^2(t)} \right) \quad (112)$$

$$D = tH - \frac{1}{2} q(t) \dot{q}(t) \quad (113)$$

$$K = t^2 H - t q(t) \dot{q}(t) + \frac{1}{2} q^2 \quad (114)$$

Inserting these expressions on the RHS of eq.(110) we get immediately the LHS. From the eqs.(112)-(114) it is also easy to see the relations between the initial conditions $(q(0), \dot{q}(0))$ and the constants (H, D_0, K_0) ; in fact:

$$H = H(t=0) = \frac{1}{2} \left(\dot{q}^2(0) + \frac{g}{q^2(0)} \right) \quad (115)$$

$$D = D_0 = -\frac{1}{2} (q(0) \dot{q}(0)) \quad (116)$$

$$K = K_0 = \frac{1}{2} q^2(0). \quad (117)$$

From the relations above we see that, inverting them, we can express $(q(0), \dot{q}(0))$ in term of (H, D_0, K_0) . The constraint (111) is already taken care by the form the (H, D_0, K_0) have in terms of $(q(0), \dot{q}(0))$.

What we want to do in this section is to see if a relation analogous to (110) exists also for our supersymmetric extension or in general if the supersymmetric system can be solved exactly. The answer is *yes* and it is based on a very simple trick.

Let us first remember eq. (78) which told us how $\widetilde{\mathcal{H}}$ and H are related:

$$i \int H(\Phi) d\theta d\bar{\theta} = \widetilde{\mathcal{H}} \quad (118)$$

The same relation holds for $\widetilde{\mathcal{D}}_0$ and $\widetilde{\mathcal{K}}_0$ with respect to D_0 and K_0 as it is clear from the explanation given in the paragraph above eqs.(45)(46):

$$i \int D_0(\Phi) d\theta d\bar{\theta} = \widetilde{\mathcal{D}}_0 \quad (119)$$

$$i \int K_0(\Phi) d\theta d\bar{\theta} = \tilde{K}_0. \quad (120)$$

Of course the same kind of relations holds for the explicitly time-dependent quantities of eqs.(90)-(93):

$$i \int D(\Phi) d\theta d\bar{\theta} = \tilde{D} \quad (121)$$

$$i \int K(\Phi) d\theta d\bar{\theta} = \tilde{K}. \quad (122)$$

Let us now build the following quantity:

$$2t^2 H(\Phi) - 4tD(\Phi) + 2K(\Phi) \quad (123)$$

This is functionally the RHS of eq.(110) with the superfield Φ^a replacing the normal phase-space variable ϕ^a . It is then clear that the following relation holds:

$$(\Phi^q)^2 = 2t^2 H(\Phi) - 4tD(\Phi) + 2K(\Phi) \quad (124)$$

The reason it holds is because, in the proof of the analogous one in q -space (eq.(110)), the only thing we used was the functional form of the (H, D, K) that was given by eq.(112)-(114). So that relation holds irrespective of the arguments, ϕ or Φ , which enter our functions provided that the functional form of them remains the same.

Let us first remember the form of Φ^q which appears⁹ on the LHS of eq.(124):

$$\Phi^q(t, \theta, \bar{\theta}) = q(t) + \theta c^q(t) + \bar{\theta} \omega^{qp} \bar{c}_p(t) + i\theta \bar{\theta} \omega^{qp} \lambda_p(t). \quad (125)$$

Let us now expand in θ and $\bar{\theta}$ the LHS and RHS of eq.(124) and compare the terms with the same power of θ and $\bar{\theta}$.

The RHS is

$$(\Phi^q)^2 = q^2(t) + \theta[2q(t)c^q(t)] + \bar{\theta}[2q(t)\bar{c}_p(t)] + \bar{\theta}\theta[2iq(t)\lambda_p(t) + 2c^q(t)\bar{c}_p(t)]. \quad (126)$$

The LHS is instead:

⁹The index $(.)^q$ is not a substitute for the index “ a ” but it indicates, as we said many times before, the first half of the indices “ a ”. Let us remember in fact that the first half of the ϕ^a are just the configurational variables q^i which in our case of a 1-dim. system is just one variable.

$$\begin{aligned}
2t^2 H(\Phi) - 4tD(\Phi) + 2K(\Phi) &= 2t^2 H(\phi) - 4tD(\phi) + 2K(\phi) + \\
&+ \theta \left[2t^2 N_H^t - 4tN_D^t + 2N_K^t \right] + \\
&- \bar{\theta} \left[2t^2 \bar{N}_H^t - 4t\bar{N}_D^t + 2\bar{N}_K^t \right] + \\
&+ i\theta\bar{\theta} \left[2t^2 \widetilde{\mathcal{H}} - 4t\widetilde{\mathcal{D}} + 2\widetilde{\mathcal{K}} \right]
\end{aligned} \tag{127}$$

where the $N_H, \bar{N}_H, N_D^t, N_K^t$ are defined in **TABLE 3** and in eqs.(95)(96), while the \bar{N}_D^t, \bar{N}_K^t are the time-dependent version¹⁰ of the operators defined in eq.(A.10). It is a simple exercise to show that all these functions ($H, D, K, N_{(\dots)}, \bar{N}_{(\dots)}, \widetilde{\mathcal{H}}, \widetilde{\mathcal{K}}, \widetilde{\mathcal{D}}$) are conserved and constants of motion in the enlarged space $(\phi^a, c^a, \lambda_a, \bar{c}_a)$.

If we now compare the RHS of eq.(126) and eq.(127) and equate terms with the same power of θ and $\bar{\theta}$, we get (by writing the N and \bar{N} explicitly):

$$q^2(t) = 2t^2 H(\phi) - 4tD(\phi) + 2K(\phi); \tag{128}$$

$$2q(t)c^a(t) = \left[2t^2 \frac{\partial H}{\partial \phi^a} - 4t \frac{\partial D}{\partial \phi^a} + 2 \frac{\partial K}{\partial \phi^a} \right] c^a; \tag{129}$$

$$2q(t)\bar{c}_p(t) = \left[2t^2 \frac{\partial H}{\partial \phi^a} - 4t \frac{\partial D}{\partial \phi^a} + 2 \frac{\partial K}{\partial \phi^a} \right] \omega^{ab} \bar{c}_b; \tag{130}$$

$$i2q(t)\lambda_p(t) + 2c^a(t)\bar{c}_p(t) = -i \left[2t^2 \widetilde{\mathcal{H}} - 4t\widetilde{\mathcal{D}} + 2\widetilde{\mathcal{K}} \right]. \tag{131}$$

We notice immediately that eq.(128) is the same as the one of the original paper [1] and solves the motion for “ q ”. Given this solution we plug it in eq.(129) and, as on the RHS we have the N -functions which are constants, once these constants are assigned we get the motion of c^a . Next we assign three constant values to the \bar{N} -functions which appear on the RHS of eq.(130), then we plug in the solution for q given by eq.(128) and so we get the trajectory for \bar{c}_p . Finally we do the same in eq.(131) and get the trajectory of λ_p .

The solution for the momentum-quantities $(p, c^p, \bar{c}_q, \lambda_q)$ can be obtained via their definition in terms of the previous variables.

The reader may be puzzled by the fact that in the space $(\phi^a, c^a, \lambda_a, \bar{c}_a)$ we have 8 variables but we have to give 12 constants of motion: $(H, D_0, K_0, N_{(\dots)}, \bar{N}_{(\dots)}, \widetilde{\mathcal{H}}, \widetilde{\mathcal{K}}, \widetilde{\mathcal{D}})$ to get the solutions from equations (128)-(131). The point is that, like in the case of the standard conformal mechanics [1], we have constraints among the constants of

¹⁰By “time-dependent version” we mean that they are related to the time independent ones in the same manner as the N^t -functions were via eqs. (95)(98).

motion. We have already one constraint and it is given by eq.(111). The others can be obtained in the following manner: let us apply the operator $c^a \partial_a$ on both sides of eq.(111) and what we get is the following relation:

$$N_H K_0 + N_K H - 2N_D D_0 = 0 \quad (132)$$

which is a constraint for the N -functions. Let us now do the same applying on both side of eq.(111) the operator $\bar{c}_a \omega^{ab} \partial_b$. What we get is:

$$\bar{N}_H K_0 + \bar{N}_K H - 2\bar{N}_D D_0 = 0 \quad (133)$$

which is a constraint among the \bar{N} -functions. Finally let us apply the Q_{BRS} on equation (133) and we will get:

$$i\tilde{\mathcal{H}}K_0 + i\tilde{\mathcal{K}}H - 2i\tilde{\mathcal{D}}D_0 - \bar{N}_H N_K - \bar{N}_K N_H + 2\bar{N}_D N_D = 0 \quad (134)$$

which is a constraints among the $\tilde{\mathcal{H}}, \tilde{\mathcal{D}}, \tilde{\mathcal{K}}$.

So we have 4 constraints (134)(133)(132)(111) which bring down the constants of motion to be specified in $(\phi^a, c^a, \lambda_a, \bar{c}_a)$ from 12 to 8 as we expected.

7 Conclusion

In this paper we have provided a new supersymmetric extension of conformal mechanics. We have realized that the model is deeply geometrical in the sense that the Grassmannian variables and the supersymmetric Hamiltonian and various other charges are all well-known objects in differential geometry. In this case it is the differential geometry of the manifold and of the flows associated to the original conformal mechanical model. We feel it was important to unveil the geometry because the recently discovered connection between conformal mechanics and black-holes or in general the *AdS/CFT* connection must have deeply geometrical origin.

What remains to be done, from a purely formal point of view, is to extend to our model the recent analysis carried out in ref. [14] where it was found that the original conformal mechanics model has a Virasoro and w_∞ algebra. We hope to come back to these issues using the tools developed in ref. [15].

Last but not least, in performing this analysis we have discovered also a new *universal* symmetry present also for non-conformal model and which, in our opinion is at the heart of the interplay classical-quantum mechanics.

Appendix

A Details of the derivation of eqs.(84)-(87)

In this appendix we are going to show the detailed calculations leading to eqs.(84)-(87). The reader may have notice the similarity between the charge Q_H (40) and the Q_D, Q_K of eqs.(47) (48). We say “similarity” because all of them are made of two pieces, the first is the Q_{BRS} for all of them. It is easy to show that also the second pieces can be put in a similar form. Like for Q_H the second piece was (see **TABLE 3**) of the form $N_H = c^a \partial_a H$, so it is easy to show that both Q_D , and Q_K can be put in the form:

$$Q_D = Q_{BRS} - 2\gamma N_D; \quad Q_K = Q_{BRS} - \alpha N_K \quad (\text{A.1})$$

where N_D and N_K are respectively:

$$N_D = c^a \partial_a D_0; \quad N_K = c^a \partial_a K_0 \quad (\text{A.2})$$

with the D_0 and K_0 given¹¹ by eqs. (17)(18). So all three (N_H, N_D, N_K) operators could be put in the general form:

$$N_X = c^a \partial_a X \quad (\text{A.3})$$

where X is either H, D_0 or K_0 . In the case of D_0 and K_0 the X is quadratic in the variables ϕ^a :

$$X = \frac{1}{2} X_{ab} \phi^a \phi^b \quad (\text{A.4})$$

where X_{ab} is a constant 2×2 matrix.

In order to find the \widehat{N}_X (that is the superspace version of N_X) we should use eqs. (79),(80) where Ω is now our operator N_X . From the expression of N_X we get for $\delta\Phi^a(t, \theta, \bar{\theta})$ of eq.(80):

$$\delta\Phi^a(t, \theta, \bar{\theta}) = \bar{\theta} \omega^{ab} (\bar{\varepsilon} \partial_b X) + i \bar{\theta} \theta \omega^{ab} (i \bar{\varepsilon} c^d \partial_d \partial_b X) \quad (\text{A.5})$$

where $\bar{\varepsilon}$ is the anticommuting parameter associated to the transformation. Given the form of X (see eq.(A.4) above), we get for (A.5):

$$\delta\Phi^a(t, \theta, \bar{\theta}) = \bar{\theta} \bar{\varepsilon} \omega^{ab} X_{bd} [\phi^d + \theta c^d]. \quad (\text{A.6})$$

¹¹Actually we should take the classical version of (17)(18) as we are doing classical mechanics.

Note that, using superfields, the above expression can be written as:

$$\delta\Phi^a(t, \theta, \bar{\theta}) = -\bar{\varepsilon}\omega^{ab}X_{bd}\bar{\theta}\Phi^d(t, \theta, \bar{\theta}). \quad (\text{A.7})$$

So we obtain from eq.(79) that the superspace expression of N_X is

$$(\widehat{N}_X)_d^a = \omega^{ab}X_{bd}\bar{\theta}. \quad (\text{A.8})$$

The same kind of analysis we did here for the Q_D and Q_K can be done also for the \overline{Q}_D and \overline{Q}_K . They can be written as:

$$\overline{Q}_D = \overline{Q}_{BRS} + 2\gamma \overline{N}_D; \quad \overline{Q}_K = \overline{Q}_{BRS} + \alpha \overline{N}_K; \quad (\text{A.9})$$

with

$$\overline{N}_D = \bar{c}_a\omega^{ab}\partial_b D; \quad \overline{N}_K = \bar{c}_a\omega^{ab}\partial_b K; \quad (\text{A.10})$$

and the superspace representation of the \overline{N}_X turns out to be:

$$(\widehat{\overline{N}}_X)_b^a = \omega^{ac}X_{cb}\theta. \quad (\text{A.11})$$

Remembering the form of the D_0 and K_0 functions in their classical version (see eq.(17)(18)) and comparing it with the general form of X of eq.(A.4) above, we get from eqs. (A.2)(A.3) that the matrices X_{ab} associated to D and K are¹² exactly those of eq.(88). So this is what we wanted to prove.

¹²We will call this form of X_{ab} as D_{ab} , and the one associated to K_0 as K_{ab} , to stick to the conventions of eqs.(84)-(87).

B Representation of H , D , K in superspace

In this appendix we will reproduce the calculations which provide the superspace representations of (H, D, K) contained in **TABLE 11**. We will start first with the operators at time $t = 0$ which are listed in eqs. (16)-(18). Using eqs.(79)(80) let us first do the variations $\delta_{(H,D,K)}\Phi^a$. As (H, D_0, K_0) contain only (ϕ^a) their action will affect only the λ_a field contained in the superfield Φ :

$$\delta_H \lambda_q = \varepsilon[H, \lambda_q] = -i\varepsilon \frac{g}{q^3} \quad (\text{B.1})$$

$$\delta_H \lambda_p = \varepsilon[H, \lambda_p] = i\varepsilon p \quad (\text{B.2})$$

$$\delta_{D_0} \lambda_q = \varepsilon[D_0, \lambda_q] = -\frac{i}{2}\varepsilon p \quad (\text{B.3})$$

$$\delta_{D_0} \lambda_p = \varepsilon[D_0, \lambda_p] = -\frac{i}{2}\varepsilon q \quad (\text{B.4})$$

$$\delta_{K_0} \lambda_q = \varepsilon[K_0, \lambda_q] = i\varepsilon q \quad (\text{B.5})$$

$$\delta_{K_0} \lambda_p = \varepsilon[K_0, \lambda_p] = 0. \quad (\text{B.6})$$

Considering that the two superfields are:

$$\Phi^q = q + \theta c^q + \bar{\theta}\bar{c}_p + i\bar{\theta}\theta\lambda_p; \quad (\text{B.7})$$

$$\Phi^p = p + \theta c^p - \bar{\theta}\bar{c}_q - i\bar{\theta}\theta\lambda_q; \quad (\text{B.8})$$

it is very easy to see that the $\hat{\Omega}$ operators on the RHS of eqs.(79) can only be the following:

$$\hat{H} = \bar{\theta}\theta \frac{\partial}{\partial t}; \quad (\text{B.9})$$

$$\hat{D}_0 = -\frac{1}{2}\bar{\theta}\theta\sigma_3; \quad (\text{B.10})$$

$$\hat{K}_0 = -\bar{\theta}\theta\sigma_-. \quad (\text{B.11})$$

Next we should pass to the representation of the time-dependent operators which are related to the time-independent ones by eqs.(89)-(91). Also for the superspace representation there will be the same relations between the two set of operators, that means:

$$\hat{H} = \hat{H}_0; \quad (\text{B.12})$$

$$\hat{D} = t\hat{H} + \hat{D}_0; \quad (\text{B.13})$$

$$\hat{K} = t^2\hat{H} + 2t\hat{D}_0 + \hat{K}_0. \quad (\text{B.14})$$

Using the above relations and the expressions obtained in eqs.(B.8)-(B.10), it is easy to reproduce the last three operators contained in **TABLE 11**.

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